Analysis of Strains from a 45° Rosette

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This document briefly describes a set of equations which can be used to recover principal strains from measurements taken with a 45° strain gauge rosette.

1 Recovery of Principal Strains

Benham *et al.* (1996) provide equations for a 45° strain gauge rosette in Chapter 11 "Stress and Strain Transformations". Figure 1 shows a configuration of planar principal strains and a rosette at a test site. Planar principal strains ϵ_1 and ϵ_2 exist at the test point, and are named so that $\epsilon_1 > \epsilon_2$. Thus, ϵ_1 is the more tensile of the strain components and ϵ_2 is the more compressive. The gauge is placed at the test site so that its ϵ_l gauge is rotated by angle θ anticlockwise from the ϵ_1 direction. Angle ϕ is defined as the anticlockwise rotation from the central gauge to the ϵ_2 direction.

Following Benham *et al.* (1996), we can then use the following equations to

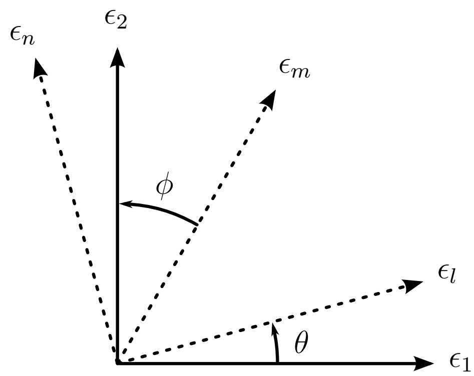

Figure 1: 45° rosette. The solid axes represent axes of principal strains. The principal compressive strains are ϵ_1 and ϵ_2, such that $\epsilon_1 > \epsilon_2$. The strain gauges of the rosette are aligned so that they measure normal strains ϵ_l, ϵ_m and ϵ_n, and are each separated by 45°. The axis of ϵ_l is rotated by an angle of θ anticlockwise from the principal tensile strain ϵ_1. Angle ϕ is defined as the anticlockwise rotation from the central gauge to the ϵ_2 direction. This is a modification of Fig 11.23 from Benham *et al.* (1996).

recover principal strains and the angle θ in this configuration

$$\epsilon_1 = \frac{1}{2}(\epsilon_l + \epsilon_n) + \frac{\sqrt{2}}{2}\sqrt{(\epsilon_l - \epsilon_m)^2 + (\epsilon_m - \epsilon_n)^2} \tag{1}$$

$$\epsilon_2 = \frac{1}{2}(\epsilon_l + \epsilon_n) - \frac{\sqrt{2}}{2}\sqrt{(\epsilon_l - \epsilon_m)^2 + (\epsilon_m - \epsilon_n)^2} \tag{2}$$

$$\theta = -\frac{1}{2}\,\mathrm{atan2}\left(\frac{2\epsilon_m - \epsilon_l - \epsilon_n}{\epsilon_l - \epsilon_n}\right) \tag{3}$$

In these equations, the $\mathrm{atan2}$ function is the two-argument arctangent function which may be found in numerous computation libraries. $\mathrm{atan2}$ is defined as follows

$$\mathrm{atan2}\,(y/x) = \begin{cases} \mathrm{sign}(y)\arctan(y/x) & \text{if } x > 0 \\ \mathrm{sign}(y)\,(\pi/2) & \text{if } x = 0 \\ \mathrm{sign}(y)\,(\pi - \arctan(y/x)) & \text{if } x < 0 \end{cases} \tag{4}$$

in which the sign function returns the sign of its argument, $\arctan$ is the single-argument arctangent, and

$$\mathrm{atan2}\,(0/x) = \begin{cases} 0 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ \pi & \text{if } x < 0 \end{cases} \tag{5}$$

Equations 1–3 have been modified to use the $\mathrm{atan2}$ function, but otherwise are identical to those given by Benham *et al.* (1996). The recovered angle, θ, is of particular interest. The use of $\mathrm{atan2}$ results in the domain of θ being $[-\pi/2, \pi/2]$, or the right-half plane. In these equations, θ always indicates the angle of the ϵ_1 axis, and never the ϵ_2 axis.

The relationship between θ and ϕ is

$$\phi + \theta + \frac{\pi}{4} = \frac{\pi}{2} \tag{6}$$

$$\phi = \frac{\pi}{4} - \theta \tag{7}$$

Hence

$$\phi = \frac{\pi}{4} + \frac{1}{2}\,\mathrm{atan2}\left(\frac{2\epsilon_m - \epsilon_l - \epsilon_n}{\epsilon_l - \epsilon_n}\right) \tag{8}$$

2 Transformation from Principal Strains to Gauge Strains

In order to check the values obtained from Equations 1, 2 and 8, it is useful to be able to reconstruct the individual gauge strains (ϵ_l, ϵ_m and ϵ_n) from the principal strains and angle relative to the central gauge (ϵ_1, ϵ_2 and ϕ). In general, strains transform as second-order tensors

$$\epsilon' = \mathbf{R}^{-1}\epsilon\mathbf{R} \tag{9}$$

where ϵ is the strain tensor before rotation

$$\epsilon = \begin{bmatrix} \epsilon_x & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_y \end{bmatrix} \tag{10}$$

ϵ' is the strain tensor after rotation

$$\epsilon' = \begin{bmatrix} \epsilon'_x & \epsilon'_{xy} \\ \epsilon'_{xy} & \epsilon'_y \end{bmatrix} \tag{11}$$

$\mathbf{R}$ is a rotation matrix, rotating clockwise by an angle θ

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{12}$$

and $\mathbf{R}^{-1}$ is the inverse rotation matrix

$$\mathbf{R}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{13}$$

The tensor shear strain, ϵ_{xy}, is related to the Engineering shear strain, γ_{xy}, as

$$\gamma_{xy} = 2\epsilon_{xy} \tag{14}$$

Expanding Equation 9 gives the equation for rotating a planar strain

$$\begin{bmatrix} \epsilon'_x \\ \epsilon'_y \\ \gamma'_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -\sin 2\theta & \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{15}$$

Using Equation 15, and realizing that for the principal strain state, $\gamma_{xy} = 0$, the following equations can be derived, relating gauge strains to principal strains and the angle ϕ

$$\epsilon_m = \epsilon_1 \sin^2 \phi + \epsilon_2 \cos^2 \phi \tag{16}$$

$$\epsilon_n = \epsilon_1 \sin^2 \left(\frac{\pi}{4} - \phi\right) + \epsilon_2 \cos^2 \left(\frac{\pi}{4} - \phi\right) \tag{17}$$

$$\epsilon_l = \epsilon_1 \sin^2 \left(\phi + \frac{\pi}{4}\right) + \epsilon_2 \cos^2 \left(\phi + \frac{\pi}{4}\right) \tag{18}$$

These equations can be used to check the results of Equations 1, 2 and 8, by taking the principal strains (ϵ_1 and ϵ_2), and the strain angle ϕ, and transforming back to the gauge strain values (ϵ_l, ϵ_m and ϵ_n).

3 Example

Consider three gauge values as follows

$$\epsilon_l = -127.7 \ \mu \tag{19}$$

$$\epsilon_m = -960.8 \ \mu \tag{20}$$

$$\epsilon_n = -572.3 \ \mu \tag{21}$$

Using Equations 1 and 2, we can obtain principal strains as follows

$$\begin{aligned}
\epsilon_1 &= \frac{1}{2}(-127.7 - 572.3) + \frac{\sqrt{2}}{2}\sqrt{(-127.7 + 960.8)^2 + (-960.8 + 572.3)^2} \\
&= 300 \ \mu \\
\epsilon_2 &= \frac{1}{2}(-127.7 - 572.3) - \frac{\sqrt{2}}{2}\sqrt{(-127.7 + 960.8)^2 + (-960.8 + 572.3)^2} \\
&= -1000 \ \mu
\end{aligned} \tag{22}$$

Using Equation 8, we can obtain the angle ϕ as follows

$$\begin{aligned}
\phi &= \frac{\pi}{4} + \frac{1}{2}\operatorname{atan2}\left(\frac{2 \times (-960.8) + 127.7 + 572.3}{-127.7 + 572.3}\right) \\
&= 10^\circ
\end{aligned} \tag{23}$$

Now, having obtained the principal strains and the angle ϕ, we can transform these back to the original strain gauge values in order to check the calculations

$$\begin{aligned}
\epsilon_m &= 300 \times \sin^2(10°) - 1000 \times \cos^2(10°) \\
&= -960.8\ \mu \\
\epsilon_n &= 300 \times \sin^2(45° - 10°) - 1000 \times \cos^2(45° - 10°) \\
&= -572.3\ \mu \\
\epsilon_l &= 300 \times \sin^2(45° + 10°) - 1000 \times \cos^2(45° + 10°) \\
&= -127.7\ \mu
\end{aligned} \tag{24}$$

References

Benham, P. P., Crawford, R. J. and Armstrong, C. G. (1996) *Mechanics of Engineering Materials (2nd ed)* Addison Wesley Longman Ltd.